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Elliptic problems with lack of compactness via a new fixed point theorem

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Abstract

This paper provides a new fixed point theorem for increasing self-mappings $G: B \to B$ of a closed ball $B \subset X$, where X is a Banach semilattice which is reflexive or has a weakly fully regular order cone X_+. By means of this fixed point theorem, we are able to establish existence results of elliptic problems with lack of compactness.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider the semilinear elliptic Dirichlet boundary value problem (BVP for short) in $W_0^{1,2}(\Omega)$ of the form

$$-\Delta u = f(x,u) + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

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where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h \in W^{-1,2}(\Omega)$. Here $W_0^{1,2}(\Omega)$ denotes the usual Sobolev space of square integrable functions having generalized homogeneous boundary values in the sense of traces, and $W^{-1,2}(\Omega)$ denotes its dual space. Problem (1.1) has been extensively investigated over the last decades by many authors, see for a example [3,5,20] and references therein. Existence results have been obtained mainly by using topological or variational methods (critical point theory) and by combining these methods with differential inequalities techniques such as the upper and lower solution method, where as a common characteristic feature the compactness of the Nemytskij operator $F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ generated by the function f and defined by

$$\langle F(u), \varphi \rangle = \int_\Omega f(x, u(x))\, \varphi(x)\, dx, \quad u, \varphi \in W_0^{1,2}(\Omega),$$

plays an important role. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$. Defining $A := -\Delta : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\langle Au, \varphi \rangle = \int_\Omega \nabla u \nabla \varphi\, dx,$$

the corresponding weak formulation of (1.1) can be written as

$$u \in W_0^{1,2}(\Omega): \quad Au - F(u) = h \text{ in } W^{-1,2}(\Omega). \tag{1.2}$$

In the theory of monotone operators due to Brezis and Browder, the compactness of F ensures the pseudomonotonicity of the operator $A - F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ which along with its coercivity yields an existence result for (1.2). The compactness of F allows also to transform problem (1.2) into a fixed point equation involving a compact fixed point operator such that the existence follows by applying Schauder's fixed point theorem provided the fixed point operator maps a closed, bounded and convex set into itself. In order to apply variational methods, a compactness condition on the energy functional associated with (1.2) is needed which guarantees the convergence of a subsequence of a minimizing sequence. This condition, which is the Palais–Smale condition, is also a consequence of the compactness of F, see for example [20]. Sufficient conditions on the function f that ensure the compactness of the Nemytskij operator F are, e.g., the following:

(i) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function,
(ii) f satisfies a subcritical growth condition: $N \geqslant 3$, $2 \leqslant p < 2^* = \frac{2N}{N-2}$
$|f(x,s)| \leqslant k(x) + c|s|^{p-1}$, $s \in \mathbb{R}$, where $k \in L_+^q(\Omega)$ with $1/p + 1/q = 1$.

Here, $2^* = \frac{2N}{N-2}$ is the critical exponent in the Sobolev embedding $W_0^{1,2}(\Omega) \subset L^p(\Omega)$. When p equals the critical Sobolev exponent 2^*, or when f is, for instance, discontinuous with respect to its second argument problem (1.1) becomes difficult because of the lack of compactness of F. Special methods such as for example the upper and lower solution method combined with truncation techniques, the dual variational principle, the concentration–compactness-principle, or nonsmooth critical point theory based on Clarke's generalized gradient have been developed to overcome the lack of compactness in the investigation of elliptic problems of the form (1.1), see e.g., [1–4,6–8,10,13,16,17,19–21,23,30,33]. For elliptic problems in unbounded domains, such as for example in all of $\mathbb{R}^N$, the situation becomes even more complicated, since the compactness, in general, is already violated for smooth enough nonlinearities f satisfying only a linear growth. This is because of the lack of compactness of the embedding $W^{1,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$. Elliptic problems in unbounded domains with nonlinearities that are discontinuous and/or of critical growth have been treated by different methods, e.g., in [9,11,14,15,18,20,22,24–26,31].

The main goal of this paper is to present an alternative approach to the existence theory of elliptic problems which allows us, in particular, to treat problems with lack of compactness without assuming the existence of an ordered pair of upper and lower solutions. Our approach is based on a general fixed point result for increasing operators in ordered normed spaces obtained recently by the second author in [27]. As a special case, we provide in Section 2 a new fixed point result for increasing self-mappings $G : B \to B$ of a closed ball $B \subset X$ of a Banach semilattice X, which is suitable in applications to elliptic problems with lack of compactness. Applications are studied in Section 3.

2. New fixed point theorem

For convenience, let us first introduce some basic concepts of partially ordered sets.

Given a nonempty partially ordered set (poset) $P = (P, \leqslant)$. The notation $x < y$ stands for $x \leqslant y$ and $x \neq y$.

An element b of P is called an *upper bound* of a subset A of P if $x \leqslant b$ for each $x \in A$. If $b \in A$, we say that b is the *maximum* of A, and denote $b = \max A$. A lower bound of A and the minimum, $\min A$, of A are defined similarly, replacing $x \leqslant b$ above by $b \leqslant x$. If the set of all upper bounds of A has the minimum, we call it a *least upper bound of* A and denote it by $\sup A$. The greatest lower bound, $\inf A$, of A is defined similarly.

We say that a poset P is a *lattice* if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in P$. A lattice P is called *complete* iff $\inf N$ and $\sup N$ exist for all nonempty subsets N of P. A subset C of a poset P is called a *chain* if $x \leqslant y$ or $y \leqslant x$ for all $x, y \in C$. We say that C is *well ordered* if each nonempty subset of C has a minimum, and *inversely well ordered* if each nonempty subset of C has a maximum. Obviously, each (inversely) well-ordered set is a chain.

We say that a sequence $(x_n)_{n=0}^{\infty}$ of a poset P is *increasing* (resp., *strictly increasing*) if $x_n \leqslant x_m$ (resp., $x_n < x_m$) whenever $n < m$. A (strictly) decreasing sequence is defined similarly, replacing $n<m$ by $m<n$. A sequence of P is called *monotone* if it is increasing or decreasing. Given posets $(P, \leqslant)$ and $(\tilde{P}, \leqslant)$ with, in general, different partial orderings $\leqslant$. We say that a mapping $G: P \to \tilde{P}$ is *increasing* iff the following is true (cf. [35]):

$$\text{for all } x, y \in P: \quad x \leqslant y \text{ implies } Gx \leqslant Gy.$$

(Notice the difference between increasing operators and *monotone* operators due to Browder and Brezis, which are not related with partial orderings, cf. [36].)

The subset X_+ of a normed space X is called an *order cone* iff the following are true:

(i) X_+ is closed, convex, and nonempty, and $X_+ \neq \{0\}$.
(ii) If $u \in X_+$ and $\alpha \geqslant 0$, then $\alpha u \in X_+$.
(iii) If $u \in X_+$ and $-u \in X_+$, then $u = 0$.

We say the order cone X_+ of a normed space X is (*weakly fully regular*) *fully regular* if each bounded and increasing sequence of X_+ is (weakly) strongly convergent.

A Banach space (normed space) $(X, \|\cdot\|)$ endowed with a partial ordering $\leqslant$ induced by an order cone X_+ by $x \leqslant y$ iff $y - x \in X_+$ is called an *ordered Banach space* (*ordered normed space*).

Definition 2.1. A Banach space $(X, \|\cdot\|)$ which is a partially ordered linear lattice satisfying, in addition, $\|x^{\pm}\| \leqslant \|x\|$ for all $x \in X$ with $x^+ = \sup\{0, x\}$ and $x^- = \inf\{0, x\}$ is called a *Banach semilattice*.

Notice that the definition of a Banach semilattice given here is a weaker one than the classical concept of Banach lattice introduced, e.g., in [12]. Special examples of Banach semilattices, which will be used later, are the L^p- and Sobolev spaces $W^{1,p}(\Omega)$. More precisely, let $\Omega \subset \mathbb{R}^N$ be a domain and let $L^p(\Omega)$ be endowed with its natural partial ordering, that is $u \leqslant w$ if and only if $w - u$ belongs to the set $L^p_+(\Omega)$ (*order cone*) of all nonnegative elements of $L^p(\Omega)$, then $\leqslant$ induces also a partial ordering in the Sobolev space $W^{1,p}(\Omega)$ and we have the following result.

Lemma 2.1. *The Banach spaces* $L^p(\Omega)$, $W^{1,p}(\Omega)$ *and* $W^{1,p}_0(\Omega)$ *are Banach semilattices for* $p \in [1, \infty)$ *which are reflexive if* $p \in (1, \infty)$, *and* $L^p_+(\Omega)$ *is a fully regular order cone of* $L^p(\Omega)$ *for* $p \in [1, \infty)$.

Lemma 2.1 readily follows from [29, Theorem 1.20] or [34, Theorem 1.56] and from results proved in [28].

Definition 2.2. Let $X = (X, \|\cdot\|, \leqslant)$ be a real ordered normed space. A subset P of X is called (*weakly*) *sequentially order compact* if monotone sequences of P have (weak) limits in P. We say that $a \in P$ is a *sup-center* of P

if $\sup\{a, y\}$ exists and belongs to P for all $y \in P$. The definition of an inf-center is analogous. P is said to have a *fixed point property* if each increasing mapping $G : P \to P$ has a fixed point.

The following fixed point result is a consequence of Theorem 1.2.1 and Proposition 1.2.1 of [28].

Lemma 2.2. *Let $P = (P, \leqslant)$ be a partially ordered set and $G : P \to P$ an increasing mapping. Then the following holds*:

(a) *For each $a \in P$ there is a unique well-ordered chain C called a w.o. chain of G-iterations of a, in P satisfying*
$a = \min C$, and if $a < x \in P$, then $x \in C$ iff $x = \sup G[\{y \in P \mid y < x\}]$.
If $a \leqslant Ga$, and if $x^* = \sup G[C]$ exists, then $x_* = \max C$, and x_* is a fixed point of G.

(b) *For each $b \in P$ there exists exactly one inversely well-ordered chain D called an i.w.o. chain of G-iterations of b, satisfying*
$b = \max D$, and if $b > x \in P$, then $x \in D$ iff $x = \inf G[\{y \in C \mid y > x\}]$. If $Gb \leqslant b$, and if $x^* = \inf G[D]$ exists, then $x^* = \min D$, and x_* is a fixed point of G.

For convenience let us recall a result from [17, Lemma A.3.1] which will be needed in the proof of our fixed point theorem.

Lemma 2.3. *Let A be a well-ordered subset of an ordered normed space X, and assume that each increasing sequence of A converges weakly in X. Then A contains an increasing sequence which converges weakly to* $\sup A$.

By means of Lemmas 2.2 and 2.3, we are in a position to prove the following fixed point theorem.

Theorem 2.1. *Let P be a weakly sequentially order compact subset of an ordered normed space X having a sup-center or an inf-center. Then P has the fixed point property.*

Proof. Assume that $a \in P$ is a sup-center of P (the proof in the case when P has an inf-center is similar), and $G : P \to P$ an increasing operator. Define $F := x \mapsto \sup\{a, Gx\}$. Since G is increasing, then also F is increasing. Let C be the w.o. chain of F-iterations of a in P. Since G is increasing, then $G[C]$ is a well-ordered chain in P. Thus, $y = \sup G[C]$ exists and belongs to P by Lemma 2.3. Because a is a sup-center of P, then $b := \sup\{a, y\}$ exists and belongs to P. It is easy to see that $b = \sup F[C]$. Since $a \leqslant Fa$, it then follows from Lemma 2.2(a) that b is a fixed point of F, and $Gb \leqslant Fb = b$. Thus, either $Gb = b$, i.e., b is a fixed point or $Gb < b$. If $Gb < b$, let D be the i.w.o. chain of G-iterations of b. Then $G[D]$ is an i.w.o. chain in $G[P]$, whence

$x^* = \inf G[D]$ exists in P by the dual of Lemma 2.3. Thus, x^* is by Lemma 2.2(b) a fixed point of G. $\square$

Remark 2.1. It can be shown that the first elements of the chain C in the proof of Theorem 2.1 are the following iterations: $x_0 = a$, $x_{n+1} = \sup\{a, Gx_n\}$, $n = 0, 1, \ldots$, as long as $x_n < x_{n+1}$. The first elements of the chain D are: $y_0 = b$, $y_{m+1} = Gy_m$, $m = 0, 1, \ldots$, as long as $y_{m+1} < y_m$. If both C and D are finite, then $b = x_n$ for some n, and $x^* = y_m$ for some m. In this case the above sequences are finite, and a fixed point of G is the last element of the sequence obtained by the following algorithm:

- $x_0 := a$: for k from 0 while $x_k \neq Gx_k$ do: $x_{k+1} := Gx_k$ if $Gx_k < x_k$ else $x_{k+1} := \sup\{a, Gx_k\}$:

This algorithm and its dual can sometimes be used to approximate fixed points of G also in cases where the chains C and/or D are not finite, as will be demonstrated by Example 3.2.

Corollary 2.1. *Let X be an ordered normed space, with (weakly) fully regular order cone X_+. Then each bounded and (weakly) closed subset P of X possessing a sup-center or an inf-center has the fixed point property.*

Proof. We only need to show that P is a weakly sequentially order compact subset of X. Let $(x_n) \subset P$ be an increasing sequence. Introduce $y_n = x_n - x_0$, $n = 1, 2, \ldots$, then $(y_n) \subset X_+$ is an increasing sequence, which is (weakly) strongly convergent because X_+ is a (weakly) fully regular order cone. Thus, the sequence (x_n) must be also (weakly) strongly convergent, and its (weak) strong limit belongs to P, since P is assumed to be a (weakly) closed subset of X. This proves the assertion. $\square$

The following result is useful in applications.

Corollary 2.2. *Let X be a Banach semilattice which is reflexive or has weakly fully regular order cone X_+. Then any closed ball $P := B_r(a) = \{x \in X \mid \|x - a\| \leqslant r\}$ of X possesses the fixed point property.*

Proof. We show first that the center a of any ball $B_r(a) = \{x \in X \mid \|x - a\| \leqslant r\}$ of the Banach semilattice X is both a sup-center and an inf-center. To this end we need to prove that for any $y \in B_r(a)$ both $\sup\{a, y\}$ and $\inf\{a, y\}$ belong to $B_r(a)$. (Notice: $\sup\{a, y\} \in X$ and $\inf\{a, y\} \in X$.) This, however, readily follows from the following representations:

$$\sup\{a, y\} = (y - a)^+ + a \quad \text{and} \quad \inf\{a, y\} = a - (a - y)^+,$$

so that by the Banach semilattice property $\|x^\pm\| \leqslant \|x\|$ we get

$$\|\sup\{a, y\} - a\| = \|(y - a)^+\| \leqslant \|y - a\| \leqslant r,$$

i.e., $\sup\{a, y\} \in B_r(a)$, and similarly $\inf\{a, y\} \in B_r(a)$, which proves the sup- and inf-center property of a. Any ball $B_r(a)$ of a Banach space is bounded, closed, and convex, and thus also weakly closed. Hence the assertion of the corollary follows from Corollary 2.1 by taking into account that if X is reflexive then its order cone X_+ is weakly fully regular. $\square$

Corollary 2.3. *Closed balls of the following spaces possess the fixed point property.*

(a) $X = (\mathbb{R}^N, \|\cdot\|_p)$ *or* $X = l^p$, $1 \leqslant p < \infty$, *ordered coordinatewise.*
(b) $X = L^p(\Omega)$, $1 \leqslant p < \infty$ ($\Omega = (\Omega, \mathscr{A}, \mu)$ *is a measure space*), *ordered a.e. pointwise.*
(c) $X = W^{1,p}(\Omega)$ *or* $X = W_0^{1,p}(\Omega)$, $1 < p < \infty$ (*as in Lemma* 2.1).

Proof. All these spaces are Banach semilattices (see also Lemma 2.1), which are reflexive for $p \in (1, \infty)$. In cases (a) and (b), the corresponding order cones X_+ are fully regular. Thus, the assertion follows from Corollary 2.2. $\square$

Remark 2.2. (i) The above fixed point theorems are formulated to meet the applications given in Section 3. For instance, the proof of Theorem 2.1 reveals that its result holds also when P is a partially ordered set having a sup-center or an inf-center, and $G : P \to P$ is an increasing mapping such that all well-ordered and inversely well-ordered chains of the range $G[P]$ of G have supremums and infimums in P.

(ii) Under the hypotheses of Corollary 2.2 the monotonicity of the operator $G : P \to P$, where $P = B_r(a)$ is a closed ball can be relaxed by the following: there is a constant $M > 0$ so that the mapping $x \to Gx + Mx$ is increasing. This is because the fixed point equation $Gx = x$ is equivalent with $\tilde{G}x = x$, where $\tilde{G}x := \frac{1}{1+M}(Gx + Mx)$ and $\tilde{G} : B_r(a) \to B_r(a)$ is increasing.

(iii) Theorem 2.1 differs from Lemma 2.2 and from many other fixed point theorems in partially ordered sets, such as, e.g., Amann's and Tarski's fixed point theorems (cf. e.g., [35, Chapter 11]) in the sense that the assumption: $a \leqslant Ga$ or $Gb \leqslant b$, is replaced by the existence of a sup-center or an inf-center of P. As shown in the proof of Corollary 2.2, this property is automatically valid in many cases important in applications.

(iv) Notice that Corollary 2.3 can be applied also when $X = L^1(\Omega)$ or $X = l^1$, even though both are not reflexive and their closed balls (with positive radius) are neither weakly nor strongly compact.

3. Application

In this section, we demonstrate the applicability of the new fixed point Theorem 2.1 in the form given by Corollary 2.2 to elliptic boundary value problems, in particular to those with lack compactness.

3.1. Elliptic problems with critical Sobolev exponent

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the semilinear elliptic Dirichlet BVP

$$-\Delta u = a\,|u|^{2^*-2}u + |u|^{p-2}u + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where $h \in W^{-1,2}(\Omega)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $1 < p < 2$, and $a > 0$ is some constant specified later. The function $u \in W_0^{1,2}(\Omega)$ is called a solution of (3.1) if for all $\varphi \in W_0^{1,2}(\Omega)$ the following relation holds:

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega (a\,|u|^{2^*-2}u + |u|^{p-2}u)\varphi \, dx + \langle h, \varphi \rangle. \tag{3.2}$$

Given a fixed $v \in W_0^{1,2}(\Omega)$, we introduce an operator F by

$$\langle F(v), \varphi \rangle = \int_\Omega (a\,|v|^{2^*-2}v + |v|^{p-2}v)\,\varphi \, dx + \langle h, \varphi \rangle, \quad \varphi \in W_0^{1,2}(\Omega).$$

Denote by $\|\cdot\|_p$, $\|\cdot\|$ and $\|\cdot\|_*$ the norms in $L^p(\Omega)$, $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$, respectively, then we obtain the estimate

$$|\langle F(v), \varphi \rangle| \leqslant a\,\|v\|_{2^*}^{2^*-1}\|\varphi\|_{2^*} + \|v\|_p^{p-1}\|\varphi\|_p + \|h\|_*\,\|\varphi\|. \tag{3.3}$$

Due to the continuous embedding $W_0^{1,2}(\Omega) \subset L^q(\Omega)$ for $1 \leqslant q \leqslant 2^*$ the operator $F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is well defined, and, moreover, the continuity and growth of the function $s \mapsto \tilde{f}(s)$ defined by $\tilde{f}(s) = a\,|s|^{2^*-2}s + |s|^{p-2}s$ imply the continuity of the operator $F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$. Let $A^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$ denote the uniquely defined solution operator of the linear Dirichlet problem

$$-\Delta u = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with $g \in W^{-1,2}(\Omega)$, then the BVP (3.1) can be written as fixed point equation

$$u \in W_0^{1,2}(\Omega) : \quad u = Gu, \tag{3.4}$$

where $G := A^{-1} \circ F : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is continuous due to the continuity of $F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ and the continuity of the operator $A^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$. Furthermore, since the function $s \mapsto \tilde{f}(s)$ is increasing, F is an increasing operator. The maximum principle implies that A^{-1} is increasing too, and thus $G : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is increasing. Let $v \in B_R \subset W_0^{1,2}(\Omega)$ where $B_R = \{v \in W_0^{1,2}(\Omega) \mid \|v\| \leqslant R\}$, and denote by c a generic positive constant. From (3.2) and (3.3), we get the following estimate:

$$c \int_\Omega (\nabla u)^2 \, dx \leqslant (a\|v\|_{2^*}^{2^*-1} + \|v\|_p^{p-1} + \|h\|_*)\,\|u\|. \tag{3.5}$$

Since $|||u||| = \int_\Omega (\nabla u)^2\, dx$ defines an equivalent norm in $W_0^{1,2}(\Omega)$, we obtain from (3.5) and in view of $u = Gv$ the estimate

$$c\|Gv\| \leqslant a\,\|v\|^{2^*-1} + \|v\|^{p-1} + \tilde{c}\,\|h\|_*. \tag{3.6}$$

From (3.6) we conclude that $G : B_R \to B_r$ is a continuous and increasing self-mapping of the Ball B_R provided there exists a positive real R satisfying the inequality

$$aR^{2^*-1} + R^{p-1} + \tilde{c}\,\|h\|_* \leqslant cR. \tag{3.7}$$

Inequality (3.7) can be satisfied by choosing R large enough such that $c\,R - R^{p-1} - \tilde{c}\,\|h\|_* > 0$, and then select the constant a sufficiently small. Thus, by applying Corollary 2.2, we obtain the following result.

Theorem 3.1. *The BVP* (3.1) *possesses solutions provided the constant a is sufficiently small.*

Remark 3.1. Due to the lack of compactness of the fixed point operator $G := A^{-1} \circ F$ Schauder's fixed point theorem cannot be applied. Since our fixed point theorem does not require continuity of the fixed point operator involved, we are able to treat also discontinuous BVP with critical growth. Problems of this kind will be treated in the following section.

3.2. Nonmonotone, discontinuous nonlinearities with critical growth

Let Ω be as in Section 3.1, and consider the BVP

$$-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.8}$$

where the nonlinearity f on the right-hand side of (3.8) may be nonmonotone, discontinuous and, in addition, may have critical growth. More precisely, we impose the following hypotheses on f:

- (H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is sup-measurable, i.e., $x \mapsto f(x, u(x))$ is measurable in Ω whenever $u : \Omega \to \mathbb{R}$ is measurable.
- (H2) $|f(x,s)| \leqslant k_1(x) + c_1\,|s|^{p_1-1}$ for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $1 < p_1 \leqslant 2^* := \frac{2N}{N-2}$, $k_1 \in L^{\frac{p_1}{p_1-1}}(\Omega)$, and $c_1 \geqslant 0$.
- (H3) The mapping $s \mapsto q(x,s) + f(x,s)$ is increasing in s for a.e. $x \in \Omega$, where
  - (q) $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $q(x,\cdot)$ is increasing, $q(\cdot, 0) = 0$ and $|q(x,s)| \leqslant k_2(x) + c_2\,|s|^{p_2-1}$ for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $k_2 \in L^{\frac{p_2}{p_2-1}}(\Omega)$, $1 < p_2 \leqslant 2^*$ and $c_2 \geqslant 0$.

We shall show that if $1 < p_1, p_2 < 2$ in (H2) and (H3), or if $p_1 = p_2 = 2$ and the constants c_1 and c_2 are small enough, or if $2 < p_1, p_2 \leqslant 2^*$ and the norms

$\|k_1\|_{\frac{p_1}{p_1-1}}$ and $\|k_2\|_{\frac{p_2}{p_2-1}}$ are sufficiently small, then the BVP (3.8) possesses at least one weak solution $u \in W_0^{1,2}(\Omega)$, i.e.

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f(x, u(x)) v(x) \, dx \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{3.9}$$

As in Section 3.1, we convert Eq. (3.9) into an equivalent operator equation of the form

$$Au = Fu \quad \text{in } W^{-1,2}(\Omega), \tag{3.10}$$

defining $A : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\langle Au, v \rangle := a(u, v) := \int_\Omega \nabla u \nabla v \, dx, \quad u, v \in W_0^{1,2}(\Omega). \tag{3.11}$$

Obviously, A is linear, bounded, and strongly monotone. By hypothesis (H1) the function f is sup-measurable, and the growth condition (H2) implies that the superposition operator $u \mapsto f(\cdot, u(\cdot))$ maps $L^{p_1}(\Omega)$ into $L^{\frac{p_1}{p_1-1}}(\Omega)$. Noticing also that the embeddings $W_0^{1,2}(\Omega) \subset L^{p_1}(\Omega)$ and $L^{\frac{p_1}{p_1-1}}(\Omega) \subset W^{-1,2}(\Omega)$ are continuous, it follows that the relation

$$\langle Fu, v \rangle = \int_\Omega f(x, u(x)) \, v(x) \, dx, \quad u, v \in W_0^{1,2}(\Omega), \tag{3.12}$$

defines an operator $F : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ (not necessarily continuous). In view of (3.11) and (3.12) a function $u \in W_0^{1,2}(\Omega)$ is a weak solution of the BVP (3.8), i.e., a solution of (3.9) if and only if u is a solution of the operator Eq. (3.10).

The properties given for the function q in (H3)(q) imply that the superposition operator $u \mapsto q(\cdot, u(\cdot)) : L^{p_2}(\Omega) \to L^{\frac{p_2}{p_2-1}}(\Omega)$ is continuous. Moreover, the embeddings $W_0^{1,2}(\Omega) \subset L^{p_2}(\Omega)$ and $L^{\frac{p_2}{p_2-1}}(\Omega) \subset W^{-1,2}(\Omega)$ are continuous. Thus, the mapping $Q : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ given by

$$\langle Qu, v \rangle = \int_\Omega q(x, u(x)) \, v(x) \, dx, \quad u, v \in W_0^{1,2}(\Omega), \tag{3.13}$$

is well-defined and continuous.

Lemma 3.1. *The operator* $A + Q : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ *is bijective, and its inverse operator* $(A + Q)^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$ *is increasing.*

Proof. Obviously the operator $A + Q : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is continuous and bounded. Since $q(x, \cdot)$ is increasing by (H3), we obtain by applying

(3.11), (3.13) and the Poincaré–Friedrichs inequality

$$\begin{aligned}
&\langle (A+Q)u-(A+Q)v, u-v\rangle \\
&\quad = \langle A(u-v), u-v\rangle + \langle Qu-Qv, u-v\rangle \\
&\quad = a(u-v,u-v) + \int_\Omega (q(x,u(x))-q(x,v(x)))(u(x)-v(x))\,dx \\
&\quad \geqslant a(u-v,u-v) = \int_\Omega |\nabla(u-v)|^2\,dx \geqslant c\|u-v\|_{1,2}^2. \qquad (3.14)
\end{aligned}$$

This shows that $A+Q$ is strongly monotone. By Browder's theorem on monotone operators (see, e.g., [36, Theorem 26.A]) the operator $A+Q$ is bijective, and the inverse operator $(A+Q)^{-1}: W^{-1,2}(\Omega)\to W_0^{1,2}(\Omega)$ is continuous (even Lipschitz continuous). It remains to show that the operator $(A+Q)^{-1}$ is also increasing. Let $h_1,\ h_2\in W^{-1,2}(\Omega)$ satisfy $h_1\leqslant h_2$, i.e.,

$$\langle h_1, v\rangle \leqslant \langle h_2, v\rangle \quad \text{for all } v\in W_0^{1,2}(\Omega)\cap L_+^2(\Omega). \qquad (3.15)$$

Denoting $u_i=(A+Q)^{-1}h_i,\ i=1,2$, it follows from (3.11), (3.13) and (3.15) that

$$\begin{aligned}
&a(u_1-u_2, v) + \int_\Omega (q(x,u_1(x))-q(x,u_2(x)))v(x)\,dx \\
&\quad = \langle (A+Q)u_1-(A+Q)u_2, v\rangle = \langle h_1-h_2, v\rangle \leqslant 0 \qquad (3.16)
\end{aligned}$$

for all $v\in W_0^{1,2}(\Omega)\cap L_+^2(\Omega)$. Taking in (3.16) the special nonnegative test function $v=(u_1-u_2)^+$, and noticing that due to the monotonicity of q the inequality

$$\int_\Omega (q(x,u_1(x))-q(x,u_2(x)))(u_1-u_2)^+(x)\,dx \geqslant 0$$

holds, and that $a((u_1-u_2)^-, (u_1-u_2)^+)=0$, we obtain from (3.14) and (3.16)

$$c\|(u_1-u_2)^+\|_{1,2}^2 \leqslant a((u_1-u_2)^+, (u_1-u_2)^+) = a(u_1-u_2, (u_1-u_2)^+) \leqslant 0.$$

This implies that $(u_1-u_2)^+=0$, i.e., $u_1\leqslant u_2$, which concludes the proof. □

As an application to Corollary 2.2 and Lemma 3.1, we shall prove the following existence result for the BVP (3.8).

Theorem 3.2. *Under hypotheses* (H1)–(H3) *the BVP* (3.8) *possesses weak solutions in the following cases*:

(a) $1<p_1,p_2<2$.
(b) $p_1=p_2=2$ *and the constants* c_1 *and* c_2 *are small enough.*
(c) $2<p_1,p_2\leqslant 2^*$ *and the norms* $\|k_1\|_{\frac{p_1}{p_1-1}}$ *and* $\|k_2\|_{\frac{p_2}{p_2-1}}$ *are sufficiently small.*

Proof. Recall that a function $u \in W_0^{1,2}(\Omega)$ is a weak solution of the BVP (3.8) if and only if u is a solution of the operator equation (3.10). The space $X = W_0^{1,2}(\Omega)$ has the properties listed in Corollary 2.2 when the partial ordering of X is defined by the order cone $L_+^2(\Omega)$ and the norm of X given by

$$\|u\|^2 = \|u\|_{1,2}^2 = \int_\Omega (|\nabla u(x)|^2 + u(x)^2)\,dx, \quad u \in X. \tag{3.17}$$

The operator $A + Q$ is bijective by Lemma 3.1. Due to the monotonicity of $s \mapsto q(x,s) + f(x,s)$ the operator $F + Q$ is increasing. Since $(A+Q)^{-1}$ is increasing by Lemma 3.1, then $G = (A+Q)^{-1} \circ (F+Q)$ as the composition of increasing operators is increasing. To find a closed ball $B_R = B_R(0)$ of X such that $G[B_R] \subseteq B_R$, let $v \in W_0^{1,2}(\Omega)$ be given. By means of (H2), (H3), (3.12) and (3.13) we obtain for each $u \in W_0^{1,2}(\Omega)$,

$$\begin{aligned}|\langle (F+Q)v, u\rangle| &\leqslant \int_\Omega (|f(x,v(x))| + |q(x,v(x))|)\,|u(x)|\,dx \\ &\leqslant (\|k_1\|_{\frac{p_1}{p_1-1}} + c_1\,\|v\|_{p_1}^{p_1-1})\|u\|_{p_1} \\ &\quad + (\|k_2\|_{\frac{p_2}{p_2-1}} + c_2\,\|v\|_{p_2}^{p_2-1})\|u\|_{p_2}.\end{aligned} \tag{3.18}$$

Since $q(\cdot,0) = 0$ by hypothesis (H3) (q), it follows from (3.14) when $v = 0$ that

$$\langle (A+Q)u, u\rangle \geqslant c\|u\|_{1,2}^2 \quad \text{for all } u \in W_0^{1,2}(\Omega). \tag{3.19}$$

By (3.17), (3.18), (3.19) and the continuous embeddings $W_0^{1,2}(\Omega) \subset L^{p_i}(\Omega)$, $i = 1,2$, we get the estimate for $u = Gv$:

$$\begin{aligned}\|u\|^2 &\leqslant \frac{1}{c}\langle (A+Q)u, u\rangle = \frac{1}{c}\langle (F+Q)v, u\rangle \\ &\leqslant d(\|k_1\|_{\frac{p_1}{p_1-1}} + \|k_2\|_{\frac{p_2}{p_2-1}} + c_1\,\|v\|_{p_1}^{p_1-1} + c_2\,\|v\|_{p_2}^{p_2-1})\|u\| \\ &\leqslant d(\|k_1\|_{\frac{p_1}{p_1-1}} + \|k_2\|_{\frac{p_2}{p_2-1}} + \tilde{c}_1\,\|v\|^{p_1-1} + \tilde{c}_2\,\|v\|^{p_2-1})\|u\|\end{aligned}$$

for some positive constant d, which implies the inequality

$$\|Gv\| = \|u\| \leqslant d\,(\|k_1\|_{\frac{p_1}{p_1-1}} + \|k_2\|_{\frac{p_2}{p_2-1}} + \tilde{c}_1\,\|v\|^{p_1-1} + \tilde{c}_2\,\|v\|^{p_2-1}).$$

Thus,

$$\begin{cases} \|Gv\| \leqslant M + \psi(\|v\|), \text{ where} \\ M = d\,(\|k_1\|_{\frac{p_1}{p_1-1}} + \|k_2\|_{\frac{p_2}{p_2-1}}) \text{ and} \\ \psi(R) = d(\tilde{c}_1\,R^{p_1-1} + \tilde{c}_2\,R^{p_2-1}). \end{cases} \tag{3.20}$$

(a) If $1 < p_1, p_2 < 2$, then $0 < p_1 - 1, p_2 - 1 < 1$, so that if ψ is given by (3.20), then $R - \psi(R) \to \infty$ as $R \to \infty$. Thus, $M + \psi(R) \leqslant R$ for R sufficiently large.

(b) If $p_1 = p_2 = 2$, then in (3.20) $\psi(R) = b\,R$, where $b = d(\tilde{c}_1 + \tilde{c}_2)$. Hence, if the constants c_1 and c_2 are so small that $b < 1$, then $M + \psi(R) \leqslant R$ if $R \geqslant \frac{M}{1-b}$.

(c) If $2 < p_1, p_2 \leqslant 2^*$, then in (3.20) $\psi(R) < R$ when R is a sufficiently small positive number. Consequently, if the norms $\|k_1\|_{\frac{p_1}{p_1-1}}$ and $\|k_2\|_{\frac{p_2}{p_2-1}}$ are small enough, the inequality $M + \psi(R) \leqslant R$ has positive solutions R.

The above proof shows that in cases (a)–(c) there exists $R > 0$ such that $M + \psi(R) \leqslant R$. Since ψ is increasing, it then follows from (3.20) that $G[B_R] \subseteq B_R$. Since G is also increasing, then G has by Corollary 2.2 a fixed point $u \in B_R$, i.e., $u = Gu = (A + Q)^{-1} \circ (F + Q)u$. Thus $(A + Q)u = (F + Q)u$, or equivalently $Au = Fu$, whence u is a weak solution of the BVP (3.8). $\square$

Remark 3.2. The hypotheses of Theorem 3.2 allow the function f to be discontinuous in all its arguments, and the functions f and q to possess critical growth. Theorem 3.2 provides an existence result for elliptic problems defined in domains $\Omega \subset \mathbb{R}^N$ with $N \geqslant 3$. By inspection of the proof of Theorem 3.2 one can easily see that it applies also to problems in dimensions $N = 1$ and 2. Moreover, since for $N = 1$ and 2 the critical exponent is $2^* = \infty$, the inequality for the exponents $p_1,\ p_2$ in case (c) of Theorem 3.1 can be replaced by $2 < p_1,\ p_2 < \infty$.

Example 3.1. Assume that $\mathbb{R}^4$ is equipped with the Euclidean norm $|\cdot|$. Choose $\Omega = \{x \in \mathbb{R}^4 \mid \frac{1}{2} < |x| < 1\}$. Given $\lambda > 0$ and $p \in L^{\frac{4}{3}}(\Omega)$, consider the BVP

$$-\Delta u(x) = \lambda(p(x) + |u(x)|^2([u(x)] - u(x))) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.21}$$

where $[z]$ denotes the greatest integer $\leqslant z \in \mathbb{R}$.

The BVP (3.21) is of the form (3.8), where

$$f(x, s) = \lambda(p(x) + |s|^2([s] - s)).$$

The nonlinearity f is discontinuous and has critical growth. One easily verifies that the hypotheses (H1)–(H3) hold with $q(x, s) = \lambda\, s^3$, whence the BVP (3.21) has by Theorem 3.2 weak solutions if λ is small enough.

Example 3.2. Consider the BVP

$$\begin{aligned} &-u''(x) = 3[2 - x] + [(u(x))^{\frac{1}{3}} - x] \text{ a.e. in } \Omega = (0, 2), \\ &u(0) = u(2) = 0. \end{aligned} \tag{3.22}$$

Problem (3.22) is of the form (3.8), where

$$q(x, s) \equiv 0 \quad \text{and} \quad f(x, s) = 3[2 - x] + [s^{\frac{1}{3}} - x]. \tag{3.23}$$

It is easy to see that the hypotheses of Theorem 3.2(a) hold, whence the BVP (3.22) has solutions. We shall now construct one solution. By elementary calculations one can show that the operator $G = A^{-1} \circ F$ can be given in explicit form by

$$Gu(x) = \frac{2-x}{2} \int_0^x tf(t, u(t))\, dt + \frac{x}{2} \int_x^2 (2-t) f(t, u(t)))\, dt, \quad x \in [0, 2]. \tag{3.24}$$

By the proof of Theorem 3.2 there exists a closed ball B_R of $W_0^{1,2}(0,2)$ such that $G[B_R] \subset B_R$. Since 0 is an inf-center of B_R, i.e., $\inf\{0, u\} \in B_R$ for each $u \in B_R$, we apply the following algorithm which is dual to that given in Remark 2.1 to approximate a fixed point of G:

$$u_0 := 0\text{: for } k \text{ from } 0 \text{ while } u_k \neq Gu_k \text{ do: if } u_k < Gu_k \text{ then}$$
$$u_{k+1} := Gu_k \text{ else } u_{k+1} := \min\{0, Gu_k\}.$$

Calculating the values of the functions u_k numerically by Simpson rule, one obtains the following estimate for a solution of (3.22):

$$u(x) \approx \begin{cases} -1.5x^2 + 1.444\, x, & 0 \leqslant x < 0.666, \\ -x^2 + 0.776x + 0.223, & 0.666 < x \leqslant 1, \\ x^2 - 3.224x + 2.223, & 1 < x \leqslant 1.334, \\ 1.5x^2 - 4.556x + 3.111, & 1.334 < x \leqslant 2. \end{cases} \tag{3.25}$$

In view of this approximation and the boundary conditions $u(0) = u(2) = 0$, one can infer that a solution of (3.22) is of the form

$$u(x) = \begin{cases} -\frac{3}{2} x^2 + a_1 x, & 0 \leqslant x \leqslant \frac{2}{3}, \\ -x^2 + a_2 x + a_3, & \frac{2}{3} \leqslant x \leqslant 1, \\ x^2 + a_4 x + a_5, & 1 \leqslant x \leqslant \frac{4}{3}, \\ \frac{3}{2} x^2 + a_6 x - 6 - 2a_6, & \frac{4}{3} \leqslant x \leqslant 2. \end{cases} \tag{3.26}$$

Because u and u' are continuous at the points $\frac{2}{3}$, 1 and $\frac{4}{3}$, we get six equations, from which one can solve for the coefficients a_i, $i = 1, \ldots, 6$. Substituting these values to (3.26) we obtain

$$u(x) = \begin{cases} -\frac{3}{2} x^2 + \frac{13}{9} x, & 0 \leqslant x \leqslant \frac{2}{3}, \\ -x^2 + \frac{7}{9} x + \frac{2}{9}, & \frac{2}{3} \leqslant x \leqslant 1, \\ x^2 - \frac{29}{9} x + \frac{20}{9}, & 1 \leqslant x \leqslant \frac{4}{3}, \\ \frac{3}{2} x^2 - \frac{41}{9} x + \frac{28}{9}, & \frac{4}{3} \leqslant x \leqslant 2. \end{cases} \tag{3.27}$$

This function u is easily seen to be a solution of the BVP (3.22).

3.3. Elliptic problems in unbounded domains

Let $N \geqslant 3$ and as before $2^* = \frac{2N}{N-2}$. We consider the following elliptic problem in all of $\mathbb{R}^N$:

$$u \in D^{1,2}(\mathbb{R}^N): \quad -\Delta u + \alpha(x)u = h(x) + f(x,u) \text{ in } \mathbb{R}^N, \tag{3.28}$$

where we denote by $X := D^{1,2}(\mathbb{R}^N)$ the space defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

Denote by $\|\cdot\|_p$ the norm of $L^p(\mathbb{R}^N)$, $1 < p < \infty$, then for any $u \in C_0^\infty(\mathbb{R}^N)$ the following inequality holds:

$$\|u\|_{2^*}^2 \leqslant \gamma \int_{\mathbb{R}^N} |\nabla u|^2\, dx = \|u\|^2, \tag{3.29}$$

for some positive constant γ (Sobolev constant) not depending on u, see, e.g., [34, Theorem 1.32]. In what follows let γ be the best Sobolev constant, see, e.g., [32]. According to inequality (3.29) $X = D^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2^*}(\mathbb{R}^N)$. In our study the nonlinearity f on the right-hand side of (3.28) may be discontinuous in u, so that compactness is violated due to the discontinuous behavior of f and due to the unboundedness of the domain. The following assumptions will be made:

(A1) $\alpha \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $h \in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$.

(A2) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is sup-measurable, $s \to f(x,s)$ is increasing, and the following growth estimate holds:
$|f(x,s)| \leqslant \beta(x)\,|s|^{r-1} + c|s|^{2^*-1}$, $s \in \mathbb{R}$, for some constant $c > 0$, where $2 < r < 2^*$ and $\beta \in L^t(\mathbb{R}^N)$ with $t = \frac{2N}{2N - rN + 2r}$ $(= \frac{2^*}{2^*-r})$.

Consider the bilinear form B defined by

$$B(u,\varphi) = \int_{\mathbb{R}^N} \alpha u \varphi\, dx, \quad u, \varphi \in X.$$

Due to the estimate

$$\int_{\mathbb{R}^N} |\alpha|\,|u|\,|\varphi|\, dx \leqslant \|\alpha\|_{\frac{N}{2}} \|u\|_{2^*} \|\varphi\|_{2^*},$$

and by the continuous embedding $X \subset L^{2^*}(\mathbb{R}^N)$ the bilinear form $B : X \times X \to \mathbb{R}$ is well defined, and satisfies

$$|B(u,\varphi)| \leqslant \gamma\, \|\alpha\|_{\frac{N}{2}} \|u\| \|\varphi\|. \tag{3.30}$$

Consider next the semilinear form C defined by

$$C(v,\varphi) = \int_{\mathbb{R}^N} (f(x,v) + h(x))\,\varphi\, dx, \quad v, \varphi \in X.$$

Then we have the following estimate:

$$|C(v,\varphi)| \leqslant (\|\beta\|_t \|v\|_{2^*}^{r-1} + c\,\|v\|_{2^*}^{2^*-1} + \|h\|_{\frac{2^*}{2^*-1}})\|\varphi\|_{2^*},$$

which implies in view of the continuous embedding $X \subset L^{2^*}(\mathbb{R}^N)$ that $C : X \times X \to \mathbb{R}$ is well defined and satisfies the estimate

$$|C(v,\varphi)| \leqslant \gamma^{\frac{1}{2}}(\|\beta\|_t\, \gamma^{\frac{r-1}{2}}\|v\|^{r-1} + c\gamma^{\frac{2^*-1}{2}}\|v\|^{2^*-1} + \|h\|_{\frac{2^*}{2^*-1}})\|\varphi\|. \tag{3.31}$$

Let $A : X \to X^*$ (X^* the dual space of X) be the operator defined by

$$\langle Au, \varphi \rangle = \int_{\mathbb{R}^N} \nabla u \, \nabla \varphi \, dx + B(u,\varphi), \quad u, \varphi \in X. \tag{3.32}$$

Lemma 3.2. *The operator $A : X \to X^*$ given by* (3.32) *is linear, bounded, and strongly monotone, and thus its inverse $A^{-1}: X^* \to X$ exists and is continuous, provided that $\gamma\,\|\alpha\|_{\frac{N}{2}} < 1$. Furthermore, $A^{-1}: X^* \to X$ is increasing.*

Proof. In view of (3.30) the linearity and boundedness is obvious, and from $\langle Au, u \rangle \geqslant (1 - \gamma\,\|\alpha\|_{\frac{N}{2}})\|u\|^2$ the strong monotonicity follows. Thus, the existence of the inverse $A^{-1}: X^* \to X$ which is continuous (even Lipschitz continuous) follows by Browder's theorem, cf. [36, Theorem 26.A]. Let us prove that $A^{-1}: X^* \to X$ is also increasing. To this end let $b_1,\ b_2 \in X^*$ satisfy $b_1 \leqslant b_2$ with respect to the dual order cone (i.e., $\langle b_1, \varphi \rangle \leqslant \langle b_2, \varphi \rangle$ for all $\varphi \in X_+$), and let $u_i = A^{-1} b_i,\ i = 1, 2$. We want to show that $u_1 \leqslant u_2$. From $A(u_1 - u_2) = b_1 - b_2$, we get with the nonnegative test function $\varphi = (u_1 - u_2)^+ \in X_+$ the following inequality:

$$\int_{\mathbb{R}^N} (\nabla(u_1 - u_2))(u_1 - u_2)^+ \, dx + \int_{\mathbb{R}^N} \alpha\,(u_1 - u_2)(u_1 - u_2)^+ \, dx \leqslant 0,$$

which implies in view of (3.30) that

$$\|(u_1 - u_2)^+\|^2 (1 - \gamma\,\|\alpha\|_{N/2}) \leqslant 0.$$

Thus $(u_1 - u_2)^+ = 0$, i.e., $u_1 \leqslant u_2$. $\square$

The semilinear form C defines an operator $F: X \to X^*$ given by

$$\langle Fu, \varphi \rangle = C(u,\varphi), \quad u, \varphi \in X, \tag{3.33}$$

which has the following properties.

Lemma 3.3. *The operator $F: X \to X^*$ given by* (3.33) *is well defined, bounded and increasing.*

Proof. The assertion readily follows from the definition (3.33) and the estimate (3.31), and taking into account that $s \to f(x,s)$ is increasing by assumption (A2). $\square$

The corresponding weak formulation of problem (3.28) reads as follows:

$$u \in X: \quad Au = Fu \text{ in } X^*. \tag{3.34}$$

Due to the properties of A and F according to Lemmas 3.1 and 3.2, respectively, problem (3.34) can be transformed into a fixed point equation of the form

$$u \in X: \quad u = A^{-1} \circ Fu, \tag{3.35}$$

where the fixed point operator $G := A^{-1} \circ F : X \to X$ is increasing, and satisfies an estimate of the form

$$(1 - \gamma \, \|\alpha\|_{\frac{N}{2}})\|Gv\| \leqslant \gamma^{\frac{1}{2}} \left(\|\beta\|_t \, \gamma^{\frac{r-1}{2}} \|v\|^{r-1} + c\gamma^{\frac{2^*-1}{2}} \|v\|^{2^*-1} + \|h\|_{\frac{2^*}{2^*-1}} \right). \tag{3.36}$$

Let $B_R \subset X$ be a ball of radius R. Then by (3.36), we have $G : B_R \to B_R$ provided that the inequality

$$\frac{\gamma^{\frac{1}{2}}}{(1 - \gamma\|\alpha\|_{\frac{N}{2}})} \left(\|\beta\|_t \, \gamma^{\frac{r-1}{2}} R^{r-1} + c\gamma^{\frac{2^*-1}{2}} R^{2^*-1} + \|h\|_{\frac{2^*}{2^*-1}} \right) \leqslant R \tag{3.37}$$

has positive solutions R, which is true for sufficiently small norm $\|h\|_{\frac{2^*}{2^*-1}}$. Hence, by applying Corollary 2.2, we obtain the following result.

Theorem 3.3. *Let* (A1), (A2) *and* $\gamma\|\alpha\|_{\frac{N}{2}} < 1$ *be satisfied. Then problem* (3.28) *admits solutions provided* $\|h\|_{\frac{2^*}{2^*-1}}$ *is sufficiently small.*

Remark 3.3. Semilinear elliptic problems have been considered here only for simplicity and in order to emphasize the main idea. Quasilinear elliptic operators such as the p-Laplacian or even nonpotential quasilinear elliptic operators of monotone type in the sense of Browder/Brezis can be taken into consideration as well. It should be noted that the fixed point results obtained in Section 2 can also be applied to deal with problems in nonreflexive Banach spaces.

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